

On modules over Laurent polynomial rings

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Abstract

A finitely generated $\mathbb{Z}[t, t^{-1}]$ -module without \mathbb{Z} -torsion and having nonzero order $\Delta(M)$ of degree d is determined by a pair of sub-lattices of \mathbb{Z}^d . Their indices are the absolute values of the leading and trailing coefficients of $\Delta(M)$. This description has applications in knot theory.

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1 Introduction.

The complement of a knot in the 3-sphere fibers over the circle if and only if its universal abelian cover is a product. In that case, the homology of the cover is a finitely generated abelian group, and the order of the homology as a $\mathbb{Z}[t, t^{-1}]$ -module—the Alexander polynomial of the knot—is monic.

We are motivated by a pair of simple questions: (1) What is the significance of the leading coefficient of the Alexander polynomial of a knot? (2) If k_1 and k_2 are two knots such that the leading coefficient of the Alexander polynomial of k_1 is larger in absolute value than that of k_2 , in what sense is k_1 further away from being fibered than k_2 ?

An elementary proposition about the structure of finitely generated modules over $\mathbb{Z}[t^{\pm 1}]$ provides answers. The proposition has proved useful in recent study of twisted Alexander invariants [7], but it has not previously appeared in the literature.

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2 Main result

We denote the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials by Λ . It is a Noetherian unique factorization domain.

Let M be a finitely generated Λ -module. It can be described as cokernel of a homomorphism

$$\Lambda^r \xrightarrow{A} \Lambda^s,$$

where A is an $r \times s$ *presentation matrix* with entries in Λ . Without loss of generality, we can assume that $r \geq s$.

Definition 2.1. The *order* of M is the greatest common divisor of the $s \times s$ minors of the matrix A .

We denote the order of M by $\Delta(M)$. It is well defined up to multiplication by units $\pm t^i$ in Λ . The order is nonzero if and only if M is a Λ -torsion module.

In the case that M has a square presentation matrix, the order $\Delta(M)$ is simply the determinant of the matrix. In general, a finitely generated Λ -torsion module M admits a square presentation matrix if and only if it has no nonzero finite submodule (see page 132 of [4]).

If M does not have a square presentation matrix, then we can pass to a quotient module that does without affecting the order. To do this, let zM be the maximal finite Λ -submodule of M . Define \bar{M} to be M/zM . One checks easily that \bar{M} has no nonzero finite submodule. Also, $\Delta(zM) = 1$ (see Theorem 3.5 of [4]). The following lemma is well known (see [6]). It implies that $\Delta(\bar{M}) = \Delta(M)$.

Lemma 2.2. *If N is a submodule of M , then*

$$\Delta(M) = \Delta(N)\Delta(M/N).$$

Free products with amalgamation are usually defined in the category of groups (see for example [5]). However, they can also be defined in the simpler category of abelian groups. If B, B' and U are abelian groups and $f : U \rightarrow B, g : U \rightarrow B'$ are homomorphisms, then we consider the external direct sum $B \oplus B'$ modulo the elements $(f(u), -g(u))$, where u ranges over a set of generators for U . We denote the resulting group by $B \oplus_U B'$ when the amalgamating maps f, g are understood.

Lemma 2.3. *When g is injective, the natural map $b \mapsto (b, 0)$ embeds B in $B \oplus_U B'$. Similarly, when f is injective, $b' \mapsto (0, b')$ is an embedding.*

Proof. Assume that $(b, 0)$ is trivial in $B \oplus_U B'$ for some $b \in B$. Then $(b, 0)$ is a linear combination of relators

$$(b, 0) = \sum_{i=1}^k (f(u_i), -g(u_i)),$$

where $u_1, \dots, u_k \in U$. The right-hand side can be rewritten as $(f(u), -g(u))$, where $u = \sum_{i=1}^k u_i$. Hence $g(u) = 0$. Since g is injective, $u = 0$, and so $f(u) = 0$ also.

The proof of the second statement is similar. \square

Free products with amalgamation can be formed using infinitely many factors in an analogous fashion. Here we consider groups of the form

$$\cdots \oplus_U B \oplus_U B \oplus_U \cdots$$

with identical amalgamations $B \xleftarrow{g} U \xrightarrow{f} B$. There is a natural automorphism μ that shifts coordinates one place to the right. It induces the structure of a Λ -module.

Proposition 2.4. *Assume that M is a finitely generated Λ -module.*

- *There exists a pair U, B of finitely generated abelian groups and monomorphisms $f, g : U \rightarrow B$ such that M is isomorphic to the infinite free product*

$$\cdots \oplus_U B \oplus_U B \oplus_U \cdots \tag{2.1}$$

with identical amalgamations $B \xleftarrow{g} U \xrightarrow{f} B$;

- *If U is generated by q elements, then $\deg \Delta(M) \leq q$.*

Corollary 2.5. *Let M be a finitely generated Λ -module. Assume that $\Delta(M) = c_0 + \cdots + c_d t^d$, with $c_0 c_d \neq 0$, and $\gcd(c_0, \dots, c_d) = 1$. Then there exist monomorphisms $f, g : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ such that:*

- *\bar{M} is isomorphic to the infinite free product*

$$\cdots \oplus_{\mathbb{Z}^d} \mathbb{Z}^d \oplus_{\mathbb{Z}^d} \mathbb{Z}^d \oplus_{\mathbb{Z}^d} \cdots$$

with identical injective amalgamations $\mathbb{Z}^d \xleftarrow{g} \mathbb{Z}^d \xrightarrow{f} \mathbb{Z}^d$;

- $\Delta(M) = \det(tg - f)$;
- $|c_0| = |\mathbb{Z}^d : f(\mathbb{Z}^d)|$ and $|c_d| = |\mathbb{Z}^d : g(\mathbb{Z}^d)|$.

Remark 2.6. It is well known that the coefficients of the Alexander polynomial of a knot are relatively prime and also palindromic. Moreover, the homology of the universal abelian cover of the knot complement is finitely generated as an abelian group if and only $|c_d|$ ($= |c_0|$) $= 1$. (See [1], for example.) In such case, the amalgamating maps f, g in the decomposition above identify adjacent factors of B . When the Alexander polynomial is not monic, the index $|c_d|$ measures the “gaps” between the factors. Corollary 2.5 makes the last statement precise.

3 Proof of Proposition 2.4 and Corollary 2.5.

The Λ -module M can be described by generators a_1, \dots, a_n and relators r_1, \dots, r_m , for some positive integers $n \leq m$. For $1 \leq i \leq n$ and $\nu \in \mathbb{Z}$, let $a_{i,\nu}$ denote the element $t^\nu a_i$. Similarly, for $1 \leq j \leq m$, let $r_{j,\nu}$ denote the relator $t^\nu r_j$ expressed in terms of the generators $a_{i,\nu}$. By multiplying at the outset each a_i by a suitable power of t , we can assume without loss of generality that $a_{i,0}$ occurs in at least one of $r_{1,0}, \dots, r_{m,0}$ but $a_{i,\nu}$ with $\nu < 0$ does not. Let ν_i be the largest second index of $a_{i,\nu}$ occurring in any of $r_{1,0}, \dots, r_{m,0}$.

Define B to be the abelian group with generators

$$a_{1,0}, \dots, a_{1,\nu_1}, \dots, a_{n,0}, \dots, a_{n,\nu_n}$$

and relators $r_{1,0}, \dots, r_{m,0}$. Let U be the free abelian group generated by

$$a_{1,0}, \dots, a_{1,\nu_1-1}, \dots, a_{n,0}, \dots, a_{n,\nu_n-1}.$$

(If some $\nu_i = 0$, then all $a_{i,\nu}$ are omitted.) Define a homomorphism $f : U \rightarrow B$ by mapping each $a_{i,\nu} \mapsto a_{i,\nu}$. Define $g : U \rightarrow B$ by $a_{i,\nu} \mapsto a_{i,\nu+1}$.

If either f or g is not injective, then we apply the following *reduction operation*: replace U by the quotient abelian group $U/(\ker f + \ker g)$. Replace B by $B/(f(\ker g) + g(\ker f))$, and f, g by the unique induced homomorphisms. If again f or g fails to be injective, repeat this operation. By the Noetherian condition for abelian groups, we obtain injective maps f, g after finitely many iterations.

Consider the free product with amalgamation

$$\cdots \oplus_U B \oplus_U B \oplus_U \cdots. \quad (3.1)$$

The amalgamation maps $B \xleftarrow{g} U \xrightarrow{f} B$ are injective as a consequence of the reduction operation.

A surjection h from M to the module described by (3.1) is easily defined: Map each generator $a_{1,0}, \dots, a_{1,\nu_1}, \dots, a_{n,0}, \dots, a_{n,\nu_n}$ to the generator with the same name in a chosen factor B . For any $\nu \in \mathbb{Z}$, map $t^\nu a_{1,0}, \dots, t^\nu a_{1,\nu_1}, \dots, t^\nu a_{n,0}, \dots, t^\nu a_{n,\nu_n}$ to the same generators but in the shifted factor $\mu^\nu(B)$. The amalgamating relations of (3.1) ensure that h is well defined. The relations that result from the reduction operation are mapped trivially, since the image under f (resp. g) of any element in the kernel of g (resp. f) is trivial in M . Hence h is an isomorphism. The first part of Proposition 2.4 is proved.

Assume that U is generated by q elements while B is described by Q generators and P relations. We take $q \leq Q$. A $(P+q) \times Q$ presentation matrix for the Λ -module M is easily constructed. Columns correspond to the generators of B . Rows correspond to the relations of B together with relations of the form $tg(u) = f(u)$, where u ranges over generators of U . Without loss of generality, we can assume that $P+q \geq Q$. The order $\Delta(M)$ is the greatest common divisor of the $Q \times Q$ minors. The greatest power of t that can occur is q . No negative powers arise. Hence $\deg \Delta(M) \leq q$.

In order to prove Corollary 2.5, we use Proposition 2.4 to write $\bar{M} = \dots \oplus_U B \oplus_U B \oplus_U \dots$ with identical injective amalgamating maps. By Lemma 2.3, B is a subgroup of \bar{M} . Since the coefficients of $\Delta(\bar{M})$ have no nontrivial common factor, \bar{M} is \mathbb{Z} -torsion free by [2]. Hence B and U are finitely generated free abelian groups. Moreover, the rank of U must equal that of B , since otherwise $\Delta(M)$ would vanish. The amalgamations $B \xleftarrow{g} U \xrightarrow{f} B$ induce a defining set of module relators $tg(u) - f(u)$, where u ranges over a basis for U . Hence the order of \bar{M} , which is equal to $\Delta(M)$, is $\det(tg - f)$. The remaining statement of Corollary 2.5 follows immediately.

4 Examples.

Some hypothesis on the finitely generated Λ -module M is needed for the conclusion of Corollary 2.5 to hold, as the next example demonstrates.

(1) Consider the cyclic Λ -module $M = \langle a \mid 2a = 0 \rangle$. Since M has a square presentation matrix, zM is trivial and hence $\bar{M} = M$. In this case, $B \cong \mathbb{Z}/2\mathbb{Z}$ and U is trivial. Since every nonzero element of M has order 2, it contains no free abelian subgroup B . Lemma 2.3 shows that the conclusion of Corollary 2.5 fails in this example.

The hypothesis of Corollary 2.5 on the coefficients of $\Delta(M)$ is not a necessary condition, as we see in the next example.

(2) Consider the cyclic Λ -module $M = \langle a \mid (2t - 2)a = 0 \rangle$. As in the previous example, $zM = 0$ and hence $\bar{M} = M$. Although the coefficients of $\Delta(M) = 2t - 2$ are not relatively prime, M can be expressed in the form (2.1) with B and U infinite cyclic groups and $f, g : U \rightarrow B$ given by $u \mapsto 2u$.

(3) Let G be a finitely presented group and $\epsilon : G \rightarrow \mathbb{Z}$ an epimorphism. Denote the kernel of ϵ by K . Choose an element x such that $\epsilon(x) = 1$. The abelianization $M = K/K'$ is a finitely generated Λ -module with $t(a + K') = xax^{-1} + K'$ for all cosets $a + K'$. The module depends only on G, ϵ and not on the choice of x .

Consider the case that $G = \pi_1(\mathbb{S}^3 \setminus k)$ is the group of an oriented knot k , the map ϵ is abelianization, and x is a meridian. The module M is the homology of the infinite cyclic cover of $\mathbb{S}^3 \setminus k$. It is well known that M is \mathbb{Z} -torsion free [3]. In particular, the conclusion of Corollary 2.5 holds.

Splitting the 3-sphere along a Seifert surface S for the knot with minimal genus g produces a relative cobordism Y between two copies S_-, S_+ of S . Let $B = H_1 Y$ and $U = H_1 S$. By Alexander duality, $U \cong B \cong \mathbb{Z}^{2g}$. Let $f : H_1 S_- \rightarrow B$ and $g : H_1 S_+ \rightarrow B$ be homomorphisms induced by inclusion. Then M can be expressed in the form (2.1). However, f, g are generally not injective. The reduction operation described in the proof of Proposition 2.4 can reduce the rank of B . In fact, when k has Alexander polynomial 1, the group B must become trivial.

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